

Nonrigid-Earth Rotation Solution

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Abstract. The accuracy of the rigid-Earth solution SMART97 is $2 \mu\text{as}$ over the time interval (1968, 2023). To obtain a nonrigid-Earth solution, we use the transfer function of Mathews (1999). The perturbations of the third component of the angular velocity vector are taken into account.

1. The rigid-Earth solution SMART97

The differential equations of the rigid-Earth rotation are

$$\begin{aligned}
 \ddot{\omega} &= \frac{L}{A} + \frac{B-A}{B} \sin \tilde{\varphi} \left(\frac{M}{A} \cos \tilde{\varphi} - \frac{L}{A} \sin \tilde{\varphi} \right) - \dot{\psi} \dot{\varphi} \sin \omega \\
 &\quad - \frac{C-B}{A} \dot{\psi} \sin \omega (\dot{\varphi} + \dot{\psi} \cos \omega) \\
 &\quad + \frac{B-A}{A} \frac{C-A-B}{B} \times (\dot{\psi} \sin \omega \sin^2 \tilde{\varphi} + \frac{1}{2} \dot{\omega} \sin 2\tilde{\varphi}) (\dot{\varphi} + \dot{\psi} \cos \omega), \\
 \sin \omega \ddot{\psi} &= \frac{M}{B} + \frac{B-A}{A} \sin \tilde{\varphi} \left(\frac{M}{B} \sin \tilde{\varphi} + \frac{L}{B} \cos \tilde{\varphi} \right) + \frac{C+B-A}{B} \dot{\varphi} \dot{\omega} \\
 &\quad + \frac{C-A-B}{B} \dot{\psi} \dot{\omega} \cos \omega \\
 &\quad + \frac{B-A}{A} \frac{C-A-B}{B} \times (\dot{\omega} \sin^2 \tilde{\varphi} - \frac{1}{2} \dot{\psi} \sin \omega \sin 2\tilde{\varphi}) (\dot{\varphi} + \dot{\psi} \cos \omega), \\
 \ddot{\varphi} &= \frac{N}{C} - \ddot{\psi} \cos \omega + \dot{\psi} \dot{\omega} \sin \omega \\
 &\quad + \frac{B-A}{C} \left(\frac{1}{2} \dot{\omega}^2 \sin 2\tilde{\varphi} - \dot{\psi} \dot{\omega} \cos 2\tilde{\varphi} \sin \omega - \frac{1}{4} \dot{\psi}^2 \sin 2\tilde{\varphi} (1 - \cos 2\omega) \right).
 \end{aligned}$$

The torque (L , M , N) has to be computed by using simultaneously the zonal and the tesseral harmonics because the first and the second derivatives of the diurnal and the semidiurnal terms are very important with respect to the long-period terms. Table 1 gives the amplitudes of the semidiurnal term (coming from $C_{2,2}$ and $S_{2,2}$), of the 18.6-year term and of the 13.66-day term.

Integrated in this way, the rigid Earth solution SMART97 (Bretagnon *et al.*, 1998) can reach a high accuracy. It has been compared with a numerical integration using DE403 (Standish *et al.*, 1995) for the positions of the Moon, the Sun and the planets. The accuracy is $2 \mu\text{as}$ over 1968–2023. Figure 1 gives the differences for ψ , ω and for the Earth rotation angle φ .

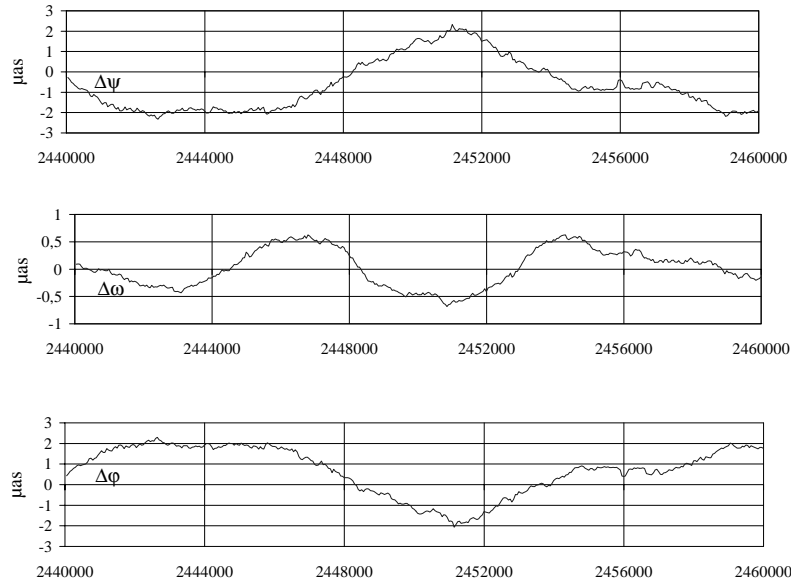


Figure 1. Theory SMART97 — numerical integration using DE403 over 1968–2023.

2. The influence of the truncation level

Table 2 gives the number of the periodic terms for $\sigma = 0.01\mu\text{as}$, $\sigma = 0.1\mu\text{as}$, and $\sigma = 1\mu\text{as}$ and the number of Poisson terms greater than σ over [J2000.0–100 yrs, J2000.0+100 yrs], for the rigid-Earth solution SMART97. The table also gives the accuracy of the solution for different levels of truncation. Data are given for the variable p ($\Delta\psi$ in the IERS Conventions 1996) of nutation in longitude reckoned from the equinox of date. The number of terms of the Herring solution for a nonrigid Earth (McCarthy, 1996) is given by comparison.

Table 1. Amplitude of the 18.6-year, 18.66-day, 12-hour terms and of their first and second derivatives.

period	$\psi(\text{in } ")$	$\dot{\psi}(\text{in } "/\text{yr})$	$\ddot{\psi}(\text{in } "/\text{yr}^2)$
18.6 years	17.280776	5.838	1.96
13.66 days	0.221507	37.212	6251.37
12 hours	0.000036	0.132	762.56

The Poisson terms In SMART97, the $P \times t$ Poisson terms (Poisson terms of degree 1) represent 13.6% of the periodic terms with a truncation of $1\mu\text{as}$, 16.8% with a truncation of $0.1\mu\text{as}$ and 20.8% with a truncation of $0.01\mu\text{as}$. That must be compared to the Herring solution in which the $P \times t$ Poisson terms represent 29.1% of the periodic terms with a truncation of $1\mu\text{as}$.

The truncation In the construction of the SMART solution for a nonrigid Earth using the transfer function of Mathews, the amplitudes of some terms in resonance can be increased considerably. So, to obtain all the terms greater than $0.1 \mu\text{as}$ in the nonrigid-Earth solution it is necessary to keep a level of truncation of $0.01 \mu\text{as}$ for the rigid-Earth solution. We give the number of terms of the SMART solution for a nonrigid Earth in Table 3.

Table 2. Number of terms of the nutation in longitude p in SMART97 for different truncation levels σ .

Solution and σ	P	$P \times t$	$P \times t^2$	$P \times t^3$	$P \times t^4$	Total	Accuracy
SMART ($0.01 \mu\text{as}$)	3910	815	183	14	2	4924	$2.2 \mu\text{as}$
SMART ($0.10 \mu\text{as}$)	1586	266	61	4	0	1917	$8 \mu\text{as}$
SMART ($1.00 \mu\text{as}$)	642	87	17	1	0	747	$40 \mu\text{as}$
Herring ($1.00 \mu\text{as}$)	375	109	0	0	0	484	

Table 3. Number of terms of the nutation in longitude of the nonrigid Earth solution at the $0.1 \mu\text{as}$ truncation level.

Periodic	$P \times t$	$P \times t^2$	$P \times t^3$	$P \times t^4$	Total
1581	264	60	4	0	1909

3. Nonrigid-Earth solution

3.1. The transfer function of Mathews (1999)

To compute the nonrigid-Earth solution, we use the transfer functions in a strict process. For instance, the transfer function of Mathews (1999) is

$$T(\sigma; e|e_R) = \frac{(e_R - \sigma)}{(e_R + 1)} N_0 \left(1 + (1 + \sigma) \sum_{\alpha} \frac{N_{\alpha}}{\sigma - \sigma_{\alpha}} \right),$$

with $e_R = 0.003284507$, $(1 + \sigma)$ is the nutation frequency in space. We use the complex frequencies

$$\begin{aligned} \sigma_1 &= (0.0025940, -0.0001438546), \sigma_2 = (-1.00231861, 0.00002578), \\ \sigma_3 &= (-0.998957, 0.000687), \sigma_4 = (0.000413499, 0.000000280), \end{aligned}$$

and the complex coefficients

$$\begin{aligned} N_0 &= (1.0000099, -0.37652854 \times 10^{-8}), \\ N_1 &= (-0.79952969, 0.043796154), \\ N_2 &= (0.048964919, 0.16332679 \times 10^{-2}), \\ N_3 &= (0.29445472 \times 10^{-3}, -0.82328898 \times 10^{-4}), \\ N_4 &= (-0.15139223 \times 10^{-4}, -0.11248592 \times 10^{-5}). \end{aligned}$$

3.2. Application to the Earth's angular velocity vector

We apply the new transfer function not to the quantities $\sin \varepsilon_0 \Delta\psi$ and $\Delta\varepsilon$ but to the Earth's angular velocity vector (p, q, r) of the rigid case expressed as a function of the three Euler angles

$$\begin{aligned} p_R &= \dot{\psi}_R \sin \omega_R \sin \varphi_R + \dot{\omega}_R \cos \varphi_R, \\ q_R &= \dot{\psi}_R \sin \omega_R \cos \varphi_R - \dot{\omega}_R \sin \varphi_R, \\ r_R &= \dot{\psi}_R \cos \omega_R + \dot{\varphi}_R. \end{aligned} \quad (1)$$

For each prograde or retrograde argument, the angular velocity vector of the nonrigid Earth is obtained by multiplication by the transfer function

$$(p_{NR} + i q_{NR}) = (p_R + i q_R) \times T(\sigma; e|e_R).$$

From the quantities (p_{NR}, q_{NR}, r_{NR}) , we can compute the derivatives of ψ , ω and φ in the nonrigid case by the inverse of (1)

$$\begin{aligned} \dot{\psi}_{NR} &= (p_{NR} \sin \varphi_{NR} + q_{NR} \cos \varphi_{NR}) / \sin \omega_{NR}, \\ \dot{\omega}_{NR} &= (p_{NR} \cos \varphi_{NR} - q_{NR} \sin \varphi_{NR}), \\ \dot{\varphi}_{NR} &= r_{NR} - \dot{\psi}_{NR} \cos \omega_{NR}. \end{aligned} \quad (2)$$

These equations are strict but the right-hand members depend on ω and φ and the derivative of ψ , and we have to proceed by iterations to solve this system. The process converges without difficulties. The precision of the computation of $\dot{\psi}_{NR}$ is better than $3.5 \times 10^{-9}''/\text{year}$ which yields an accuracy of $0.01 \mu\text{as}$ for the 18.6-year term and $0.50 \mu\text{as}$ for the 883 year term. *The classical method* is to assume $r_{NR} = r_R$. The results are illustrated by the following table. We denote SM97M99 as the solution obtained by SMART97 + Mathews (1999). We can see the results from Mathews's function are very close to the series of Herring (McCarthy, 1996).

Solution	Argument	p (sin)	p (cos)	ε (sin)	ε (cos)
SM97M99	$\lambda_3 + D - F$	17 206 664	-3 357	-1 488	-9 205 156
Herring		17 206 394	-3 702	-1 523	-9 205 474
SM97M99	$2\lambda_3$	-1 318 625	-662	-486	573 040
Herring		-1 318 526	-670	-471	573 046
SM97M99	$2\lambda_3 + 2D$	-227 663	309	150	97 854
Herring		-227 720	269	136	97 864
SM97M99	λ_3	-36 674	-123 068	16 616	684
Herring		-36 777	-123 010	16 590	698

The full method is to take into account the tidal variations in the Earth's rotation. We introduced the series $(\omega - \omega_S)$ and $(\omega - \omega_D)$ (McCarthy, 1996) in order to obtain r_{NR} from r_R ,

$$r_{NR} = r_R + (\omega - \omega_S) + (\omega - \omega_D).$$

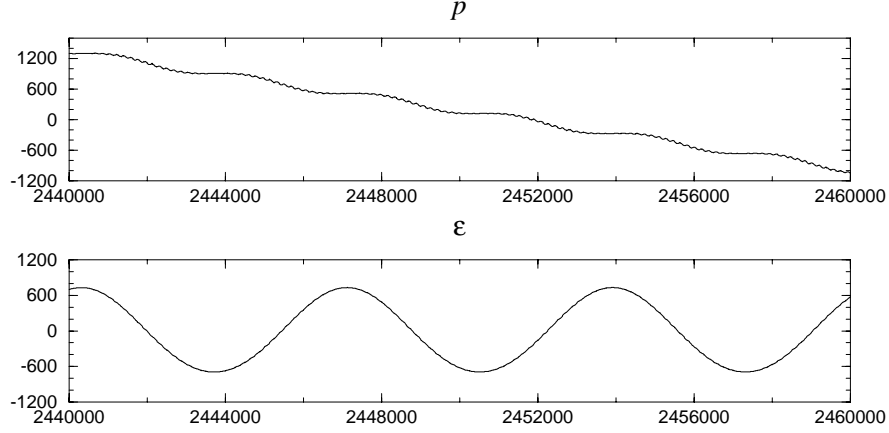


Figure 2. Influences of $(\omega - \omega_S)$ and $(\omega - \omega_D)$ on p and ε over 1968–2023. Units are μas .

The modifications are very important not only for the third Euler angle φ but also for the first two. For instance, we obtain

$$\begin{aligned}\Delta p &= 42\,242\,t + 69 \sin(2\lambda_3 + 2D - 2F) - 4 \sin(\lambda_3 + D - F) + \dots, \\ \Delta \varepsilon &= -15\,t + 712 \cos(\lambda_3 + D - F) - 19 \cos(2\lambda_3 + 2D - 2F) + \dots,\end{aligned}$$

where the amplitudes are in μas and the time in thousands of Julian years from J2000. Therefore, for ε , the 18.6-year term becomes

$$\varepsilon = -1\,488 \sin(\lambda_3 + D - F) - 9\,204\,444 \cos(\lambda_3 + D - F).$$

By comparison with the previous table we see the necessity to determine again the transfer function. The modifications of p and ε are plotted in Figure 2.

4. Conclusion

We have now to determine a new nonrigid-Earth rotation solution by using the new model of Mathews *et al.* (2000) taking into account the tidal variations in the Earth's rotation.

References

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